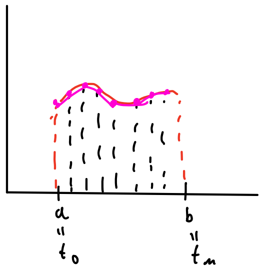
LUNGHEZZA CURVA

$f : [a, b] \to \mathbb{R}$ $\quad\quad \ell(\gamma)$ = LUNGHEZZA DI γ



POLIGONARE P

- SO CALCOLARE $\ell(P)$ = APPROSIMAZIONE PER DIFETTO DI $\ell(\gamma$

DEF: $\ell(\gamma) = \sup \ell(P)$

↳ γ È RETTIFICABILE QUANDO $\sup \ell(P) < +\infty$

↳ γ NON È RETTIFICABILE QUANDO $\sup \ell(P) = +\infty$

- SUPPONIAMO CHE γ AMMETTA UNA RAPPRESENTAZIONE PARAMETRICA ATTRAVERSO $\bar{v}(t)$ $\boxed{\text{REGOLARE}}$ CON $t \in I$

1. $t \to \bar{v}(t)$ È $C^1(I)$
2. $\bar{v}'(t) \neq \bar{0}$ $\forall t \in I$

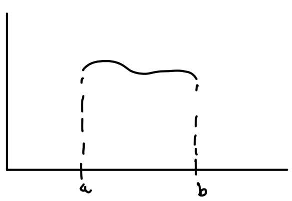
- TH.

SE $\bar{v}(t)$ È REGOLARE ALLORA γ È RETTIFICABILE
E $\ell(\gamma) = \int_a^b \| \bar{v}'(t) \| \, dt$

$I = [a, b]$

ESEMPIO GENERALE

CASO IN CUI γ È IL GRAFICO DI UNA FUNZIONE $f : [a, b] \to \mathbb{R}$



$f \in C^1[a, b]$

CURVA NON CHIUSA, SEMPLICE E REGOLARE

1) $\bar{v}'(t) \in C^1[a, b]$? $\to$ VERO

$$\bar{v}(t) = \begin{cases} x(t) = t \\ y(t) = f(t) \end{cases}$$

2) $(1, f'(t)) \neq \bar{0} \neq (0, 0) \to$ VERO

$$\ell(\gamma) = \int_a^b \sqrt{x'(t)^2 + y'(t)^2} = \int_a^b \sqrt{1 + (f'(t))^2} \, dt$$

ESEMPIO

$f(x) = \frac{2}{3}(x-1)^{3/2}$ $\quad\quad x \in [1, 4]$

$f'(x) = (x-1)^{1/2} = \sqrt{x-1}$ $\quad\quad f \in C^1[1, 4]$

$$l(\gamma) = \int_1^4 \sqrt{x + t - x}\, dt$$

PRIMITIVA DI $\sqrt{t}$

$$= \int_1^4 \sqrt{t}\, dt = \frac{2}{3} t^{3/2} \Big|_1^4$$

$$= \frac{2}{3}(8-1) = \frac{14}{3}$$

ESEMPIO

$f(x) = \ln(1-x^2)$ $\quad t \in [a,b]$ con $-1 < a < b < 1$

$$f'(x) = \frac{1}{1-x^2}(-2x) \in [a,b] \Rightarrow f \in C^1[a,b]$$

$$l(\gamma) = \int_a^b \sqrt{1 + \frac{4t^2}{(1-t^2)^2}}\, dt = \int_a^b \sqrt{\frac{(1-t^2)^2 + 4t^2}{(1-t^2)^2}}$$

$$= \int_a^b \frac{\sqrt{1+t^4-2t^2+4t^2}}{1-t^2}$$

$$= \int_a^b \frac{\sqrt{1+t^4+2t^2}}{1-t^2}\, dt$$

$$= \int_a^b \frac{\sqrt{(t^2+1)^2}}{1-t^2}\, dt$$

$$= \int_a^b \frac{1+t^2}{1-t^2}\, dt = \int_a^b \frac{2}{1-t^2}\, dt - \int_a^b 1\, dt$$

$$\frac{1+t^2}{1-t^2} = \frac{2}{1-t^2} - 1 = \frac{2-1+t^2}{1-t^2}$$

ALLORA:

$$l(\gamma) = \int_a^b \frac{1}{1-t}\, dt + \int_a^b \frac{1}{1+t}\, dt - \int_a^b 1\, dt$$

γ $j(t)$ NON È IL GRAFICO DI UNA $f(x)$ ESPLICITA

ESEMPIO

1) $\begin{cases} x(t) = t^3 \\ y(t) = t^2 \end{cases}$

$\gamma'(t) = \begin{cases} x'(t) = 3t^2 \\ y'(t) = 2t \end{cases}$ $\quad \bar{\gamma}'(0) \neq \bar{0} \to$ NON È REG. IN $t=0$
PERÒ DATO CHE È UN ESTREMO POSSO UTILIZZARE TH.

$$\ell(\gamma) = \int_0^1 \|\vec{v}'(t)\|\, dt = \int_0^1 \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

$$= \int_0^1 \sqrt{9t^4 + 4t^2}\, dt = \frac{1}{18}\int_0^1 18t\sqrt{9t^2+4}\, dt$$

FORMULARIO

$$\int f^{\alpha}(x) * f'(x)\, dx = \frac{f^{1+\alpha}(x)}{1+\alpha} + c \quad \text{con } \alpha \neq -1$$

$f(t) = 9t^2 + 4 \qquad \alpha = 1/2$

$f'(t) = 18t \quad \to$ impongo $f'(t) = 18t$

$$= \frac{1}{18} \left. \frac{(9t^2+4)^{3/2}}{3/2} \right|_0^1 = \frac{1}{27}\left[13^{3/2} - 8\right] = \ell(\gamma)$$

ESEMPIO

$$\vec{v}(t) = \begin{cases} x(t) = t \sin t \\ y(t) = t \cos t \end{cases} \qquad t \in [0, 2]$$

$$\vec{v}'(t) = \begin{cases} x'(t) = \sin t + t\cos t \\ y'(t) = \cos t - t \sin t \end{cases}$$

$$\ell(\gamma) = \int_0^2 \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

$$= \int_0^2 \sqrt{\sin^2 t + t^2\cos^2 t + \cos^2 t + t^2 \sin^2 t}\, dt$$

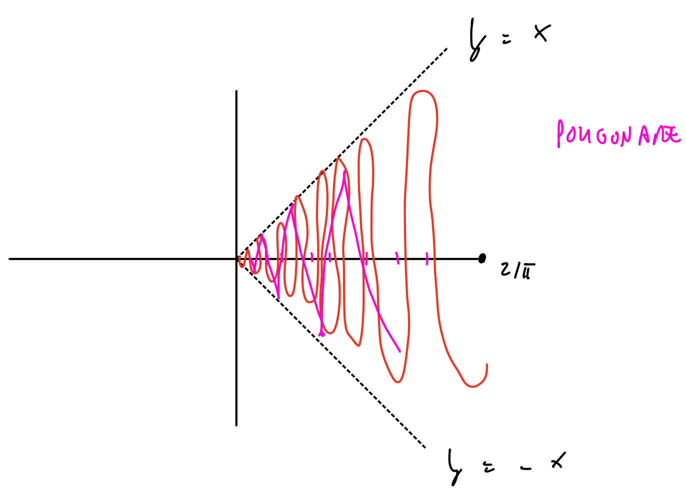
$$= \int_0^2 \sqrt{1+t^2}\, dt$$

$$\int \sqrt{1+t^2}\, dt = \frac{1}{2}\ln\left(t + \sqrt{1+t^2}\right) + \frac{1}{2} t\sqrt{1+t^2} \quad \Rightarrow \text{FORMULARIO}$$

ESEMPIO γ NON RETTIFICABILE

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$f: [0, \frac{2}{\pi}] \to \mathbb{R}$



$t_0 = 0 \qquad t_m = \frac{2}{\pi}$

$t_k = \frac{2}{k\pi}$ con $1 \leq k \leq m$

$$\| \bar{v}(t_k) - \bar{v}(t_{k-1}) \| = \sqrt{(x(t_k) - x(t_{k-1}))^2 + (f(t_k) - f(t_{k-1}))^2} \geq |f(t_k) - f(t_{k-1})|$$

$f(x) = x \sin\left(\frac{1}{x}\right)$

k PARI
$k-1$ DISPARI

$k = 2r$
$k-1 = 2r-1$

$$f(t_k) = t_k \cdot \sin\left(\frac{\pi}{2} k\right) = t_k \sin(\pi r) = 0$$

$$f(t_{k-1}) = t_{k-1} \sin\left(\frac{\pi}{2} k\right) \Rightarrow t_{k-1} \sin\left(\frac{\pi}{2}(2r-1)\right) = \pm t_{k-1}$$

$$|f(t_k) - f(t_{k-1})| = + t_{k-1}$$

$$\ell(P) = \sum_1^m \| \bar{v}(t_k) - \bar{v}(t_{k-1}) \| \geq$$

$$\geq \sum_1^m \frac{2}{\pi} \frac{1}{k-1} = \frac{2}{\pi} \sum_1^m k \frac{1}{k-1}$$

$$\sup \ell(P) \geq \frac{2}{\pi} \sum_1^{\infty} m \frac{1}{m-1} = +\infty$$

INTEGRALE DI LINEA DI 1ª SPECIE

CURVA γ $\qquad I = [a,b]$

$\bar{v} : [a,b] \to \mathbb{R}^2 \quad \gamma = \bar{v}(I)$

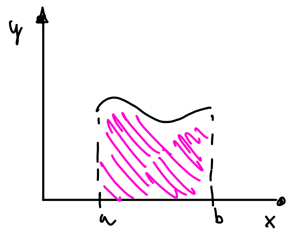
$f : A \subseteq \mathbb{R}^2 \to \mathbb{R} \qquad \gamma \subseteq A \qquad f$ CONT. SEMPRE

TH.

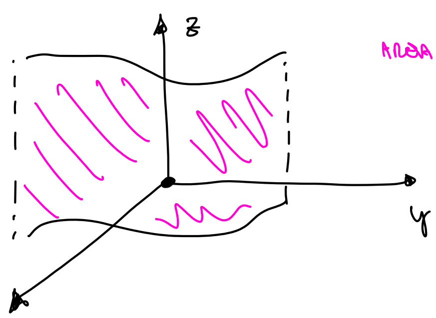
- SE $\bar{v}$ È REGOLARE ALLORA L'INTEGRALE DI LINEA SU f:

$$\int_\gamma f \, ds = \int_a^b f(\bar{v}(t)) \cdot \| \bar{v}'(t) \| \, dt$$

$f : [a,b] \to \mathbb{R}$ CONTINUA



$$\int_a^b f(x) \, dx = \text{AREA TRAPEZOIDE}$$

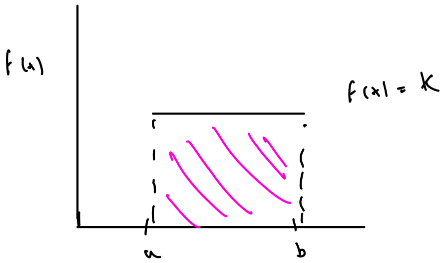


AREA $= \int_\gamma f \, ds$

DEFORMATO CON BASE ≠ SEGMENTO

||

CURVA



ESEMPIO $\mathbb{R}^3$

$$f(x, y, z) = 3x - y + z$$

$$\gamma \quad \bar{v}(t) = \begin{cases} x(t) = 3t \\ y(t) = 4t - 1 \\ z(t) = t + 5 \end{cases} \qquad t \in [0, 1]$$

$$\int_\gamma f \, ds = \int_0^1 f(\bar{v}(t)) \cdot \| \bar{v}'(t) \| \, dt$$

$$\bar{v}'(t) = \begin{cases} x'(t) = 3 \\ y'(t) = 4 \\ z'(t) = 1 \end{cases} \qquad \| \bar{v}'(t) \| = \sqrt{9 + 16 + 1} = \sqrt{26}$$

$$\int_0^1 \left[3x(t) - y(t) + z(t) \right] \cdot \sqrt{26} \, dt$$

$$= \int_0^1 \left[9t - 4t + 1 + t + 5 \right] \sqrt{26} \, dt$$

$$= \sqrt{26} \int_0^1 (6t + 6) \, dt = \sqrt{26} \left(3t^2 \Big|_0^1 + 6t \Big|_0^1 \right)$$

ESERCIZIO DA ESAME

$$\bar{v}(t) = \begin{cases} x(t) = 4 \cos\left(\frac{\pi}{4} t\right) \\ y(t) = -2 \sin\left(\frac{\pi}{4} t\right) \end{cases} \qquad t \in [1, 9]$$

È REGOLARE → APPLICO TH.

DIAMO PER SCONTATO

$$f(x, y) = \frac{y}{\sqrt{1 - \frac{3}{64} x^2}} \quad \rightarrow \quad A \supseteq \gamma = \bar{v}(I)$$

$$\int_\gamma f \, ds = \int_1^9 f(\bar{v}(t)) \cdot \| \bar{v}'(t) \| \, dt$$

$$\bar{v}'(t) = \begin{cases} x'(t) = -\pi \sin\left(\frac{\pi}{4} t\right) \\ y'(t) = -\frac{\pi}{2} \cos\left(\frac{\pi}{4} t\right) \end{cases}$$

$$\|\bar{v}'(t)\| = \sqrt{x'(t)^2 + y'(t)^2}$$

$$= \sqrt{\pi^2 \sin^2\left(\frac{\pi}{4}t\right) + \frac{\pi^2}{4}\cos^2\left(\frac{\pi}{4}t\right)}$$

$$= \pi\sqrt{\sin^2\left(\frac{\pi}{4}t\right) + \frac{1}{4}\cos^2\left(\frac{\pi}{4}t\right)}$$

$$f(\bar{v}(t)) = f(x(t), y(t))$$

$$= \frac{-2\sin\left(\frac{\pi}{4}t\right)}{\sqrt{1 - \frac{3}{64}16\cos^2\left(\frac{\pi}{4}t\right)}}$$

Allora

$$\int_1^9 \frac{-2\sin\left(\frac{\pi}{4}t\right)}{\sqrt{1-\frac{3}{4}\cos^2\left(\frac{\pi}{4}t\right)}} \cdot \pi\sqrt{\sin^2\left(\frac{\pi}{4}t\right) + \frac{\cos^2\left(\frac{\pi}{4}t\right)}{4}}\, dt$$

Nota: $\sin^2(\alpha) = 1-\cos^2(\alpha$

$$\int_1^9 \frac{-2\sin\left(\frac{\pi}{4}t\right)}{\cancel{\sqrt{1-\frac{3}{4}\cos^2\left(\frac{\pi}{4}t\right)}}} \cdot \pi\sqrt{1-\cos^2\left(\frac{\pi}{4}t\right) + \frac{1}{4}\cos^2\left(\frac{\pi}{4}t\right)}$$

$$\Downarrow$$

$$\cancel{\sqrt{1-\frac{3}{4}\cos^2\left(\frac{\pi}{4}t\right)}}$$

$$-2\pi\int_1^9 \sin\left(\frac{\pi}{4}t\right) dt = \ldots ?$$